

Calculus I: Chapter I (continued)

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1.6 LIMITS

1.6.1 LIMITS OF FUNCTIONS

Calculus was created to describe how quantities change.

It is based on the fundamental concept of the *limit* of a function.

It is this idea of limit that distinguishes calculus from algebra, geometry, and trigonometry, which are useful for describing static situations.

1.6.1 Limit of functions

Example 1: Consider the function $f(x) = \frac{\sin x}{x}$, $x \neq 0$

x	$\frac{\sin x}{x}$
1	0.841471
0.1	0.998334
0.01	0.999983

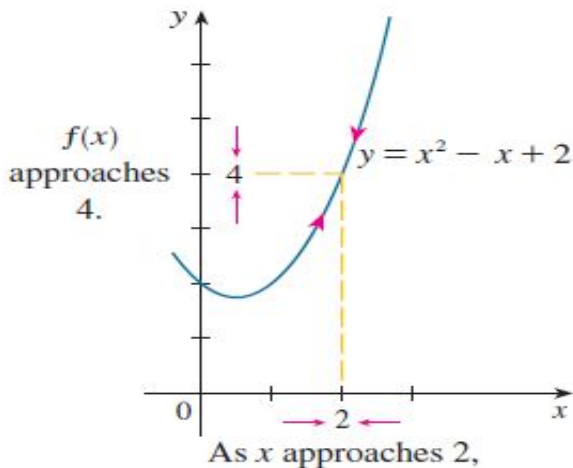
Although the function $(\sin x)/x$ is not defined at zero, as x becomes closer and closer to zero, $(\sin x)/x$ becomes arbitrarily close to 1. It is said that "the limit of $(\sin x)/x$ as x approaches zero equals 1."

1.6.1 Limit of functions

Example 2: Consider the function $f(x) = x^2 - x + 2$

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

1.6.1 Limit of functions



The limit of $f(x) = x^2 - x + 2$ as x approaches 2 equals 4.

1.6 LIMITS

1.6.1 LIMITS OF FUNCTIONS

Definition 6.1

We write

$$\lim_{x \rightarrow a} f(x) = L$$

and say **“the limit of $f(x)$ as x approaches a equals L ”** if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a . We also say that $f(x)$ **approaches L** or **converges to L** as x **approaches a** .

An alternative notation for $\lim_{x \rightarrow a} f(x) = L$ is

$$f(x) \rightarrow L \quad \text{as} \quad x \rightarrow a$$

which is usually read **“ $f(x)$ approaches L as x approaches a ”**.

1.6 LIMITS

1.6.1 LIMITS OF FUNCTIONS

In defining the limit of $f(x)$ as x approaches a , we never consider $x = a$. The value $f(a)$ itself, which may or may not be defined, play no role in the limit.

Example 6.2

(a) $\lim_{x \rightarrow a} x = a$

(b) $\lim_{x \rightarrow a} c = c$ (where c is a constant).

Example 6.3 Investigate $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1}$.

Example 6.3 Investigate $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1}$.

$x < 1$	$f(x)$
0.5	0.6666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.4000000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

Thus,

$$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = 0.5.$$

1.6 LIMITS

1.6.2 ONE-SIDED LIMITS

The limit we have discussed so far are **two-sided**.

In some instances, $f(x)$ may approach L from one side of a without necessarily approaching it from the other side, or $f(x)$ may be defined on only one side of a .

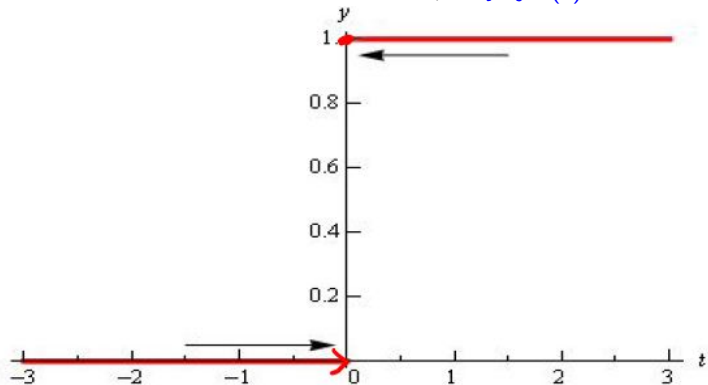
Example 6.4 The **Heaviside** function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. There is no single number that $H(t)$ approaches as t approaches 0. Therefore, $\lim_{t \rightarrow 0} H(t)$ does not exist.

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Thus, $\lim_{t \rightarrow 0} H(t)$ does not exist.



1.6 LIMITS

1.6.2 ONE-SIDED LIMITS

Definition 6.2

We write

$$\lim_{x \rightarrow a^-} f(x) = L$$

and say **“the left-hand limit of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the left) equals L ”** if we can make the values of $f(x)$ as close to L as we want by taking x to be sufficiently close to a and x less than a . We also say that $f(x)$ has **left limit** L at $x = a$.

1.6 LIMITS

1.6.2 ONE-SIDED LIMITS

Definition 6.2

We write

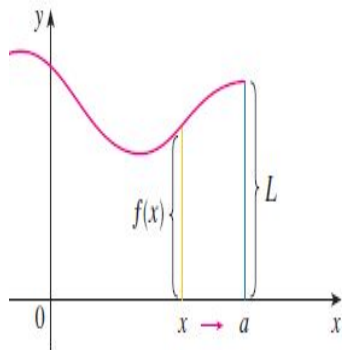
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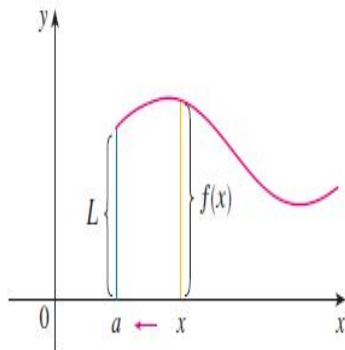
Similarly, if we require that x be greater than a , we get **“the right-hand limit of $f(x)$ as x approaches a is equal to L ”** (or $f(x)$ has **right limit** L at $x = a$), and we write

$$\lim_{x \rightarrow a^+} f(x) = L.$$

ONE-SIDED LIMITS



$$(a) \lim_{x \rightarrow a^-} f(x) = L$$



$$(b) \lim_{x \rightarrow a^+} f(x) = L$$

1.6 LIMITS

1.6.2 ONE-SIDED LIMITS

Theorem 6.1

A function $f(x)$ has limit L at $x = a$ if and only if it has both left and right limits there and these one-sided limits are both equal to L :

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L.$$

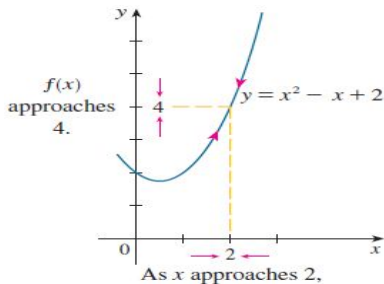
1.6 LIMITS

1.6.2 ONE-SIDED LIMITS

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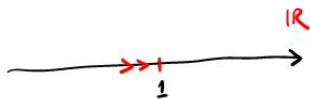
$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L.$$



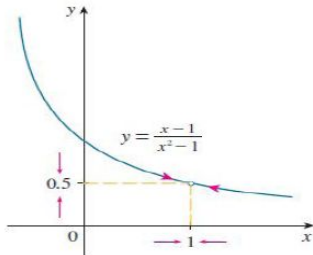
Since
 $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 4$, it follows that $\lim_{x \rightarrow 2} f(x) = 4$

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
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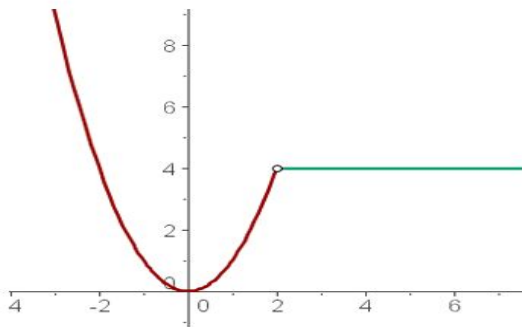


$$\lim_{x \rightarrow 1^-} \frac{x-1}{x^2-1} = 0.5 = \lim_{x \rightarrow 1^+} \frac{x-1}{x^2-1}$$



$$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = 0.5.$$

$$f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x > 2 \end{cases}$$



- $\lim_{x \rightarrow 2^-} x^2 = 4$

- $\lim_{x \rightarrow 2^+} 4 = 4$

Thus, $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = 4 = \lim_{x \rightarrow 2} f(x)$

1.6 LIMITS

1.6.2 ONE-SIDED LIMITS

Example 6.5 If

$$f(x) = \frac{|x - 2|}{x^2 + x - 6},$$

find $\lim_{x \rightarrow 2^+} f(x)$, $\lim_{x \rightarrow 2^-} f(x)$, and $\lim_{x \rightarrow 2} f(x)$.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$,
provided the limits exist.

Question What limits does $g(x) = \sqrt{1 - x^2}$ have at $x = -1$ and $x = 1$?

1.7 LAWS OF LIMITS. EVALUATING LIMITS

1.7.1 LAWS OF LIMITS

In this section, we will use the limit laws to calculate limits.

Theorem 7.1

Suppose that c is a constant and the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

$$1. \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

$$4. \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0.$$

1.7 LAWS OF LIMITS. EVALUATING LIMITS

1.7.1 LAWS OF LIMITS

6. $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$,
where n is a positive integer.

7. $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$,
where n is a positive integer.

The Limit Laws also hold for one-sided limits.

Example 7.1 Find

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}.$$

1.7 LAWS OF LIMITS. EVALUATING LIMITS

1.7.2 THE SQUEEZE THEOREM

Theorem 7.2

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

Theorem 7.3 (The Squeeze Theorem)

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$, then

$$\lim_{x \rightarrow a} g(x) = L.$$

1.7 LAWS OF LIMITS. EVALUATING LIMITS

1.7.2 THE SQUEEZE THEOREM

Theorem 7.2

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

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Theorem 7.3 (The Squeeze Theorem)

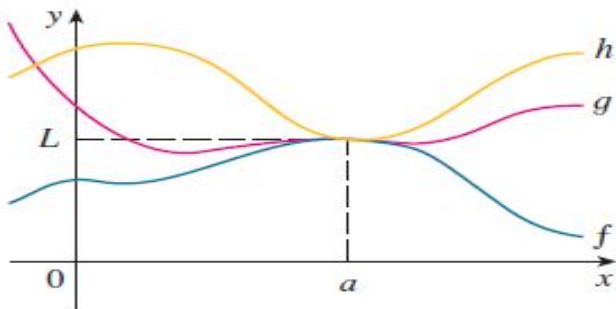
If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$, then

$$\lim_{x \rightarrow a} g(x) = L.$$

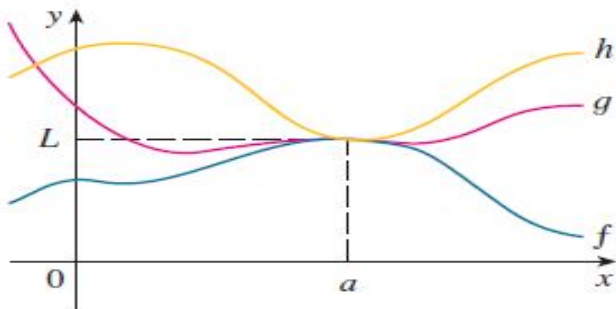
Similar statements hold for left and right limits.

The Squeeze Theorem is sometimes called the **Sandwich Theorem** or the **Pinching Theorem**.

(The Squeeze Theorem)



(The Squeeze Theorem)

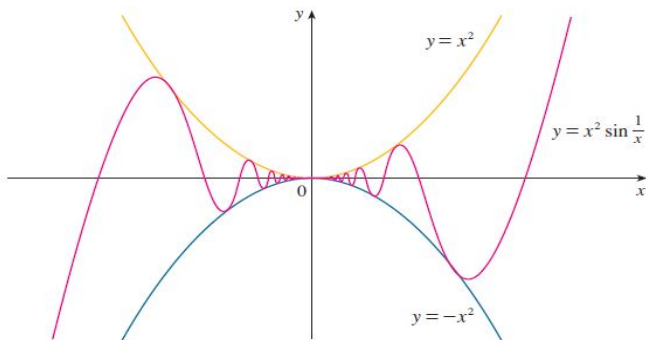


Corollary 7.1

If $\lim_{x \rightarrow a} |f(x)| = 0$, then $\lim_{x \rightarrow a} f(x) = 0$.

1.7.2 THE SQUEEZE THEOREM

Example 7.2 Show that $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$.



Theorem 7.4

If θ is measured in radians, then

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Theorem 7.4

If θ is measured in radians, then

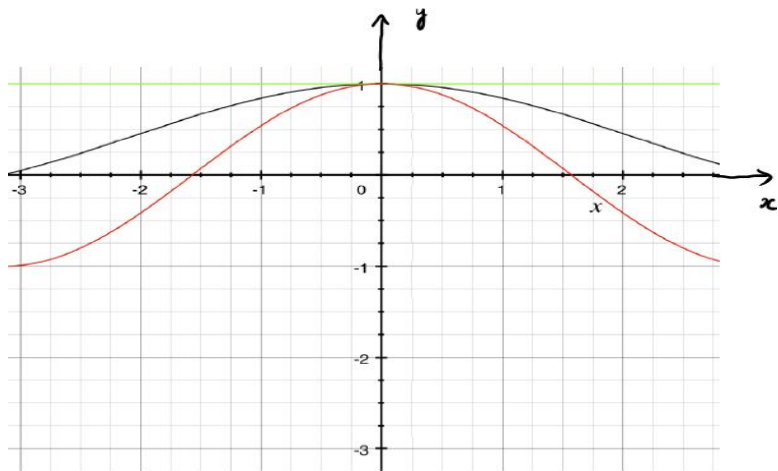
$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Proof: Note that

$$\cos x < \frac{\sin x}{x} < 1,$$

for x close enough, but not equal to 0. By the squeeze theorem, we have

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$



$$y = \cos(x)$$

$$y = \frac{\sin(x)}{x}$$

$$y = 1$$

Example: Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that $\lim_{x \rightarrow 0} f(x) = 0$ ($\mathbb{Q}$ is the set of all rational numbers).

Example: Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that $\lim_{x \rightarrow 0} f(x) = 0$ ($\mathbb{Q}$ is the set of all rational numbers).

Solution: Note that

$$0 \leq f(x) \leq x^2, \quad \forall x \in \mathbb{R}.$$

It follows from the squeeze theorem that

$$\lim_{x \rightarrow 0} f(x) = 0.$$

1.8 CONTINUITY

THE INTERMEDIATE VALUE THEOREM

1.8.1 CONTINUITY

Definition 8.1

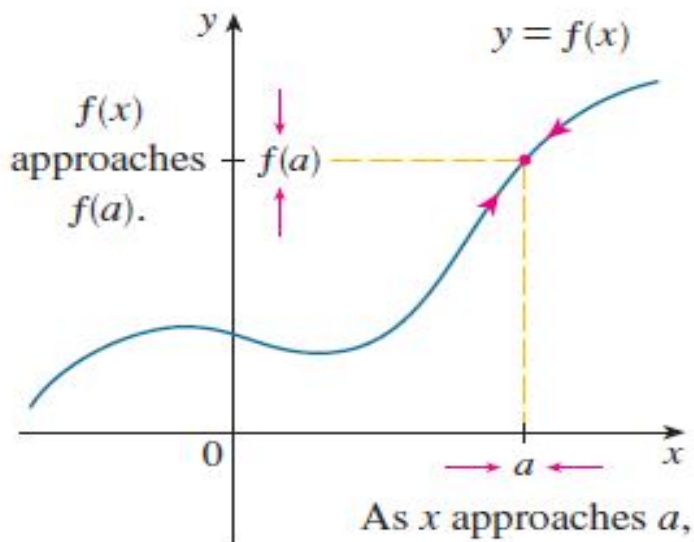
A function f is **continuous at a number a** if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

If f is not continuous at a , we say that f is **discontinuous at a** , or f has a **discontinuity at a** .

Notice that if f is continuous at a , then:

1. $f(a)$ is defined, that is, a is in the domain of f ;
2. $\lim_{x \rightarrow a} f(x)$ exists;
3. $\lim_{x \rightarrow a} f(x) = f(a)$.



1.8 CONTINUITY

THE INTERMEDIATE VALUE THEOREM

1.8.1 CONTINUITY

The definition states that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a .

- Thus, a continuous function f has the property that a small change in x produces only a small change in $f(x)$.
- In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

1.8 CONTINUITY

THE INTERMEDIATE VALUE THEOREM

1.8.1 CONTINUITY

Definition 8.2

A function f is **continuous from the right at a number a** if

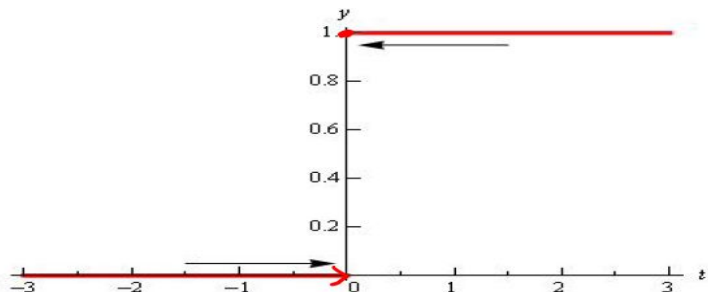
$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \rightarrow a^-} f(x) = f(a).$$

Example:

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



The Heaviside function H is **right continuous** at 0 but is not **left continuous** there.

1.8 CONTINUITY

THE INTERMEDIATE VALUE THEOREM

1.8.1 CONTINUITY

Definition 8.3

A function f is **continuous on an interval** if it is continuous at every number in the interval. If f is continuous at all points in its domain, then f is simply called **continuous**.

Here, if f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean “continuous from the right” or “continuous from the left.”

Example 8.1 Show that the function $f(x) = \sqrt{1 - x^2}$ is continuous.

1.8 CONTINUITY

THE INTERMEDIATE VALUE THEOREM

1.8.1 CONTINUITY

Theorem 8.1

If f and g are continuous at a , and c is a constant, then the following functions are also continuous at a :

- | | |
|---------------|------------------------------|
| (a) $f \pm g$ | (b) cf |
| (c) fg | (d) f/g if $g(a) \neq 0$. |

Corollary 8.1

- (a) *Any polynomial is continuous everywhere, that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.*
- (b) *Any rational function is continuous wherever it is defined-that is, it is continuous on its domain.*

1.8 CONTINUITY

THE INTERMEDIATE VALUE THEOREM

1.8.1 CONTINUITY

Example 8.2 Where are each of the following functions discontinuous?

$$(a) f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

$$(b) g(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$$(c) h(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases}$$

- The kind of discontinuity illustrated in part (a) is called **removable**.
- The discontinuity in part (b) is called an **infinite discontinuity**.
- The discontinuities in part (c) are called **jump discontinuities**.

1.8 CONTINUITY

THE INTERMEDIATE VALUE THEOREM

1.8.1 CONTINUITY

Theorem 8.2

The following types of functions are continuous at every number in their domains: Polynomials, Rational functions, Root functions, Trigonometric functions.

Example 8.3 Where is the function

$$f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$$

continuous?

1.8 CONTINUITY

THE INTERMEDIATE VALUE THEOREM

1.8.1 CONTINUITY

Theorem 8.3

If f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then $\lim_{x \rightarrow a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right).$$

Theorem 8.4

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a .

1.8 CONTINUITY

THE INTERMEDIATE VALUE THEOREM

1.8.1 CONTINUITY

Theorem 8.5

If f is continuous on an interval I with range J and if the inverse f^{-1} exists, then f^{-1} is continuous on the domain J .

1.8 CONTINUITY

THE INTERMEDIATE VALUE THEOREM

1.8.1 THE INTERMEDIATE VALUE THEOREM

Theorem 8.6 (The Intermediate Value Theorem)

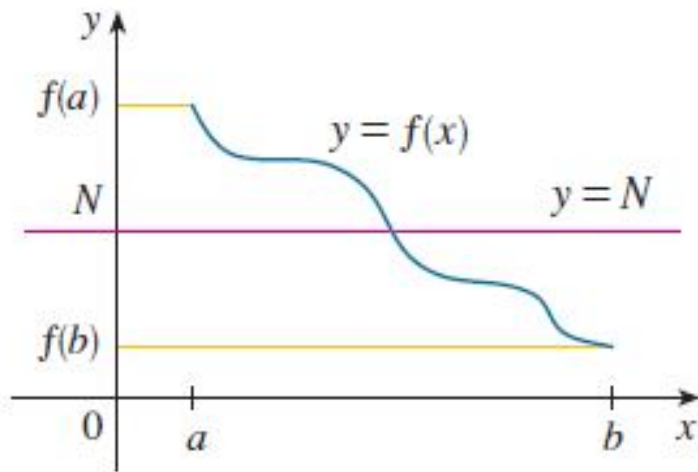
A function f that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

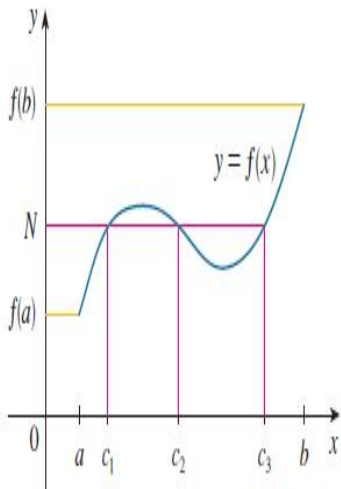
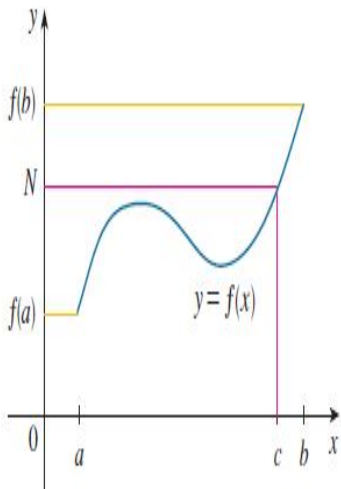
A point c where $f(c) = 0$ is called a **zero** or **root** of f .

Corollary 8.2 (Existence of Zeros)

If f is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs, that is, $f(a)f(b) < 0$, then f has a zero in (a, b) .

Example 8.4 Show that the equation $x^3 - x - 1 = 0$ has a solution in the interval $[1, 2]$.





1.9 LIMITS INVOLVING INFINITY

1.9.1 INFINITE LIMITS

Definition 9.1

Let f be a function defined on both sides of a , except possibly at a itself. Then,

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large-as large as we please-by taking x sufficiently close to a , but not equal to a .

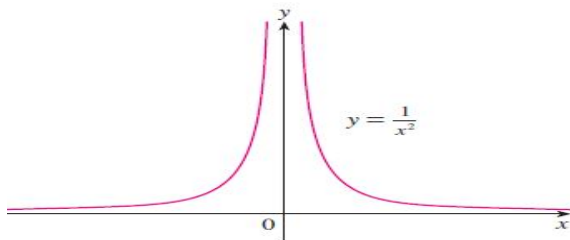
Another notation for $\lim_{x \rightarrow a} f(x) = \infty$ is:

$$f(x) \rightarrow \infty \quad \text{as} \quad x \rightarrow a.$$

Example:

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$



1.9 LIMITS INVOLVING INFINITY

1.9.1 INFINITE LIMITS

Definition 9.2

Let f be defined on both sides of a , except possibly at a itself. Then,

$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a , but not equal to a .

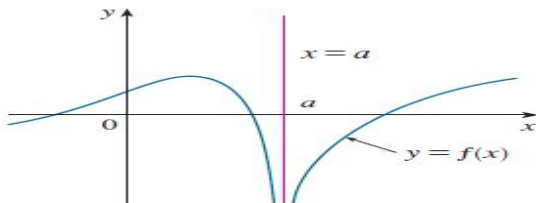


FIGURE 3

$$\lim_{x \rightarrow a} f(x) = -\infty$$

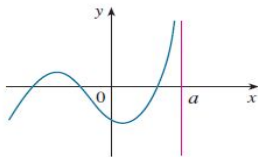
Similar definitions can be given for the one-sided limits:

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

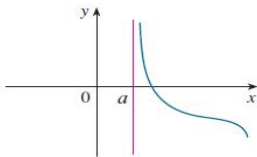
$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

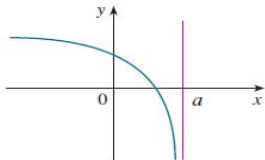
$$\lim_{x \rightarrow a^+} f(x) = -\infty$$



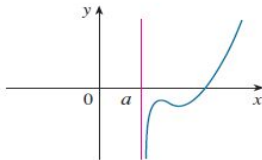
(a) $\lim_{x \rightarrow a^-} f(x) = \infty$



(b) $\lim_{x \rightarrow a^+} f(x) = \infty$



(c) $\lim_{x \rightarrow a^-} f(x) = -\infty$



(d) $\lim_{x \rightarrow a^+} f(x) = -\infty$

1.9 LIMITS INVOLVING INFINITY

1.9.1 INFINITE LIMITS

- “ $x \rightarrow a^-$ ” means that we consider only values of x that are less than a .
- “ $x \rightarrow a^+$ ” means that we consider only values of x that are greater than a .
- Keep in mind that ∞ and $-\infty$ are not numbers.

1.9 LIMITS INVOLVING INFINITY

1.9.1 INFINITE LIMITS

2 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \rightarrow a} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a} f(x) = -\infty$$

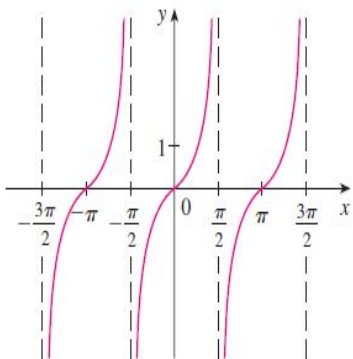
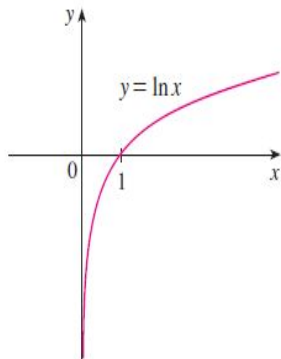
$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

Example 9.1 Find the vertical asymptotes of

(a) $f(x) = \ln x$

(b) $g(x) = \tan x$.



1.9 LIMITS INVOLVING INFINITY

1.9.2 FINITE LIMITS AT INFINITY

Now we investigate limits at infinity, where x becomes arbitrarily large, positive or negative.

Definition 9.3

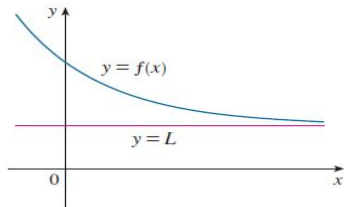
Let f be a function defined on some interval (a, ∞) . Then,

$$\lim_{x \rightarrow \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Another notation for $\lim_{x \rightarrow \infty} f(x) = L$ is

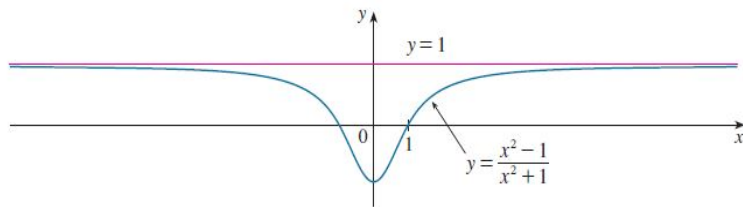
$$f(x) \rightarrow L \quad \text{as } x \rightarrow \infty.$$



1.9 LIMITS INVOLVING INFINITY

1.9.2 FINITE LIMITS AT INFINITY

Example: $\lim_{x \rightarrow \infty} \frac{x^2-1}{x^2+1} = 1$



x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

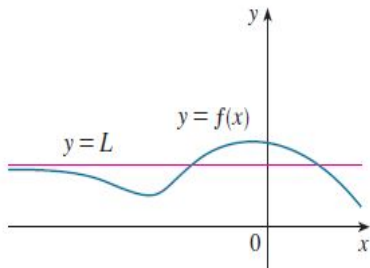
Definition 9.4

Let f be a function defined on some interval $(-\infty, a)$. Then,

$$\lim_{x \rightarrow -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

$$f(x) \rightarrow L \quad \text{as} \quad x \rightarrow \infty.$$



Definition 9.5

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L.$$

- Most of the Limit Laws given in Section 1.7 also hold for limits at infinity.

Example:

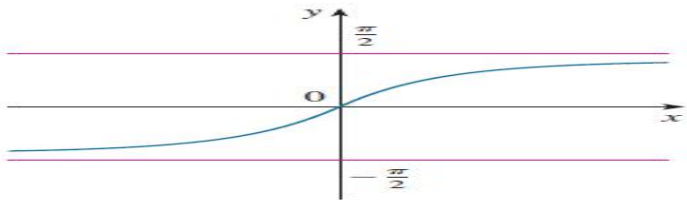


FIGURE 11
 $y = \tan^{-1} x$

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad \lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$

1.9 LIMITS INVOLVING INFINITY

1.9.2 FINITE LIMITS AT INFINITY

Example 9.2 If n is a positive integer, then

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^n} = 0.$$

Example 9.3 Evaluate

$$\lim_{x \rightarrow \pm\infty} \frac{20x^2 - 3x}{3x^5 - 4x^2 + 5}.$$

Example 9.4 Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}.$$

1.9 LIMITS INVOLVING INFINITY

1.9.2 FINITE LIMITS AT INFINITY

Theorem 9.1

The limits

$$\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x \quad \text{and} \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

exist and equal. This value is called the number e .

Thus we have

$$\lim_{t \rightarrow 0} (1 + t)^{\frac{1}{t}} = e$$

$$\lim_{t \rightarrow 0} \frac{\ln(1 + t)}{t} = 1$$

and

$$\lim_{u \rightarrow 0} \frac{e^u - 1}{u} = 1$$

1.9 LIMITS INVOLVING INFINITY

1.9.3 INFINITE LIMITS AT INFINITY

The notation

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large.

Similar meanings are attached to the following symbols:

$$\lim_{x \rightarrow \infty} f(x) = -\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = \infty$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

1.9 LIMITS INVOLVING INFINITY

1.9.3 INFINITE LIMITS AT INFINITY

Example 9.5 If $a > 1$ then

$$\lim_{x \rightarrow -\infty} a^x = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} a^x = \infty$$

Example 9.6 Calculate

(a) $\lim_{x \rightarrow \pm\infty} \frac{11x+2}{x^3-1}$

(b) $\lim_{x \rightarrow \infty} \frac{-4x^3+7x}{2x^2-3x-10}$

(c) $\lim_{x \rightarrow -\infty} \frac{-4x^3+7x}{2x^2-3x-10}$

1.9 LIMITS INVOLVING INFINITY

1.9.3 INFINITE LIMITS AT INFINITY

Asymptotic Behavior of a Rational Function The asymptotic behavior of a rational function depends only on the leading terms of its numerator and denominator. Suppose $a_n, b_m \neq 0$ and

$$L = \lim_{x \rightarrow \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0}.$$

- If $m > n$, then $L = 0$.
- If $m = n$, then $L = a_n/b_m$.
- If $m < n$, then $L = \pm\infty$, depending on the signs of numerator and denominator.

Exercises and Assignments

Text book: J. Stewart, *Calculus. Concepts and Contexts*, 2nd, Thomson Learning, 2001.

Pages	Exercises	Assignments
22–24	5, 6, 26, 35 , 40	7, 8, 18, 20, 27, 28 39 43, 48, 51
35–38	1, 10, 13	2, 4, 6, 9, 12, 14
46–49	3, 7, 24, 32, 37	5, 23, 38, 40, 44, 53, 55
73–75	5, 20, 26, 11	6, 7, 22, 25, 28, 33
86–87		19, 23, 24

Exercises and Assignments

Text book: J. Stewart, *Calculus. Concepts and Contexts*, 2nd, Thomson Learning, 2001.

Pages	Exercises	Assignments
108-110	3, 6	5, 10
117-119	2, 15, 18, 26, 32, 36	7, 10, 20, 27, 28, 33, 35, 38, 43
128-130	3, 13, 24, 29, 37	4, 6, 7, 15, 16, 23, 30 31, 33, 38, 40, 46
139-142	2, 4, 24, 39, 41	3, 11, 19, 20, 23, 31, 34 37, 42